PII: S0011-2275(98)00043-5

Cryogenics 38 (1998) 729–736

0011-2275/98/$—see front matter

Thermodynamic optimization of conduction-cooled HTS current leads

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Received 4 February 1998

A theoretical optimization is performed for the conduction-cooling method of high Tc superconductor (HTS) current leads, which can be applied to the superconducting systems cooled directly by cryogenic refrigerators without liquid helium. The current lead is a series combination of a normal metal conductor at the warmer part and a HTS at the colder part, and is cooled by a contact with distributed or staged refrigerators instead of boil-off helium gas. An analytical method is developed to derive a mathematical expression for the required refrigerator power. By incorporating the critical characteristics of the HTS, it is demonstrated that there exist unique optimal values for the current density of HTS and the joint temperature of the two parts to minimize the total refrigerator power per unit current, for a given length of the HTS. As results of the study, the absolute minimum in the refrigerator power per unit current is presented as a thermodynamic limit and the leads cooled by a two-stage refrigerator are theoretically optimized. Some aspects in practical design are also discussed with a new and useful graphical method.

Keywords: A. High Tc superconductors; B. thermodynamics; F. current leads

Nomenclature

A	Cross-sectional area of current lead, cm^2
FOM	Figure of merit of refrigerator
I	Current, A
J	Current density, A/cm^2
J_C	Critical current density, A/cm^2
J_{C0}	Critical current density at 0 K, A/cm^2
L	Length of current lead, cm
L_0	Lorentz number, W-Ω/m^2
k	Thermal conductivity, W/m-K
Q	Heat transfer rate or heat current, W
Q_{gen}	Heat generation rate, W
Q_{ref}	Cooling rate by refrigerator, W
T	Temperature, K
T_C	Critical temperature, K
U	Unit step function
W_{ref}	Refrigerator, power, W

Greek letters

λ	Lagrange multiplier
ρ	Electrical resistivity, Ω-m
τ	Dummy variable in definite integral

Subscripts

1	HTS part of current lead
2	Metal part of current lead
H	Warm end
J	Joint of HTS and metal parts
L	Cold end
min	Minimum
opt	Optimum

High Tc superconductors (HTS) are the best materials to minimize refrigeration power for the current leads in superconducting systems, mainly because they are perfect electrical conductors and have much lower thermal conductivity than the normal metals. During the past several years, a number of studies[1–8] have been performed to prove successfully that the cooling load or the electrical power for the refrigeration of the leads could be reduced significantly

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by replacing the lower temperature part of the conventional metallic lead with the HTS current leads. This so-called binary or hybrid current lead is a series combination of a normal metal conductor as a high temperature part and HTS as a lower temperature part. The cooling method for the binary leads could be quiet different from the standard helium-vapor-cooling of the conventional metallic lead, depending on how the liquid cryogens and/or the cryocoolers are employed[1].

Recent progress in the development of 4 K Gifford-McMahon refrigerators[9] has raised the possibility of the liquid-free or the refrigerator-cooled superconducting magnets[10,11]. Since there is no liquid cryogen in those superconducting systems, the current leads should be conduction-cooled in vacuum by contact with refrigerators. The heat would then be removed from the lead at one or more intermediate axial locations as well as at the cold end. The present authors think that the feasibility of the conduction-cooled HTS current lead has been demonstrated by the recent construction and operation of several prototypes[10,11] and the next crucial step towards the practical application could be the development of energy-efficient current leads.

The cooling of the HTS leads without the boil-off helium gas has been partly considered in some of the previous publications[1–8]. Most of these research works are, however, related to the design or the analysis for the HTS leads whose ends were cooled by liquid nitrogen or liquid helium and still might not provide enough information on the optimal cooling scheme for the liquid-free HTS leads. For conventional metallic leads, the conduction-cooling method was examined and completely optimized by Hilal[12]. In that theoretical work, Hilal showed by the method of calculus of variations that the refrigerator power could reach an absolute minimum with optimally distributed Carnot refrigerators and optimally sized leads. A few years before Hilal's work, Bejan and Smith[13] derived an absolute minimum of the refrigerator power required to cool a given geometry of mechanical supports for cryogenic apparatus. In a thermal point of view, the mechanical supports are quite similar to the HTS current leads that do not generate heat in a superconducting state.

The present study investigates the optimal conditions in the conduction-cooling method for the binary current leads, by combining the two optimization methods mentioned above. The first thermodynamic optimization includes the distribution of the refrigeration along the axis of a lead and the dimensions of the lead to minimize the refrigeration work as a thermodynamic limit. In addition, the optimization for a two-stage conduction-cooling is emphasized because of its practical importance.

Absolute minimum of refrigerator power

The most efficient conduction-cooling of a binary current is schematically shown in *Figure 1*. The binary lead is composed of an HTS part (denoted by subscript 1) on the cold end and a normal metal part (denoted by subscript 2) on the warm end. It is assumed that an infinite number of Carnot or reversible refrigerators are distributed along the axis of the lead and remove heat by conduction. The heat current through the lead from the warm to the cold end, Q, is defined as positive. The heat at the cold end and the cooling load at the joint are denoted by Q_L and Q_J, respectively. Q_J is the difference between the heat from the metal at the

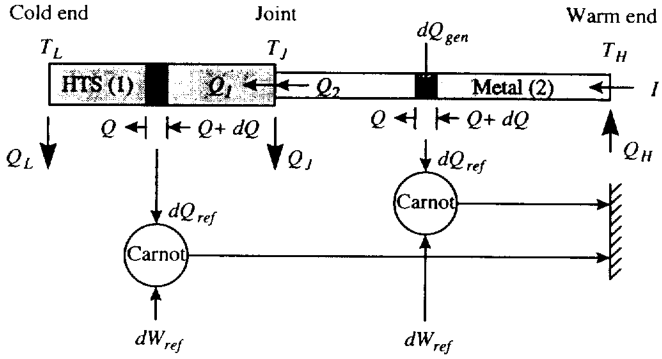

Figure 1 Binary current lead cooled by distributed Carnot refrigerators

joint, Q_2, and the heat to the HTS at the joint, Q_1. In order to consider a thermodynamic limit, it is further assumed that there is no heat generation in the HTS or at its contact surfaces and that the end temperatures are fixed.

The total power required for the distributed refrigeration can be expressed as

$$W_{ref} = \left(\frac{T_H}{T_L} - 1\right)\cdot Q_L + \int_{T_L}^{T_J}\left(\frac{T_H}{T} - 1\right)\cdot dQ_{ref} + \left(\frac{T_H}{T_J} - 1\right)\cdot Q_J + \int_{T_J}^{T_H}\left(\frac{T_H}{T} - 1\right)\cdot dQ_{ref} \quad (1)$$

where T_L, T_J and T_H are the temperatures at the cold end, the joint and the warm end, respectively, and T_H is assumed to be identical to room-temperature at which the reversible refrigerators reject heat.

Since the refrigerator should remove the increment in the heat current for an infinitesimal length of the HTS lead as shown in *Figure 1* or $dQ_{ref} = dQ$, the second term of the right-handed side in Equation (1) can be integrated by parts.

$$\int_{T_L}^{T_J}\left(\frac{T_H}{T} - 1\right)\cdot dQ_{ref} = \left(\frac{T_H}{T_J} - 1\right)\cdot Q_1 - \left(\frac{T_H}{T_L} - 1\right)\cdot Q_L + T_H\int_{T_L}^{T_J}\frac{Q}{T^2}\, dT \quad (2)$$

For an infinitesimal length of the metal lead shown in *Figure 1*, the energy balance equation can be written as $dQ_{ref} = dQ + dQ_{gen}$, where dQ_{gen} is the heat generation rate over the length. By combining the one-dimensional equations for the Fourier's heat conduction and the Ohm's heat generation, it can be simply shown that

$$dQ_{gen} = \frac{\rho_2 k_2 I^2}{Q}\, dT \quad (3)$$

where ρ and k are the electrical resistivity and the thermal conductivity, respectively and I is the current that the lead is carrying. The fourth term in Equation (1) is similarly integrated by parts, with the energy balance equation and Equation (3).

$$\int_{T_J}^{T_H}\left(\frac{T_H}{T}-1\right)\cdot dQ_{ref}=-\left(\frac{T_H}{T_J}-1\right)\cdot Q_2+T_H\int_{T_J}^{T_H}\frac{Q}{T^2}\,dT \tag{4}$$

$$+\int_{T_J}^{T_H}\left(\frac{T_H}{T}-1\right)\frac{\rho_2 k_2 I^2}{Q}\,dT$$

Substituting Equations (2) and (4) into Equation (1),

$$W_{ref}=T_H\int_{T_L}^{T_H}\frac{Q}{T^2}\,dT+\int_{T_J}^{T_H}\left(\frac{T_H}{T}-1\right)\frac{\rho_2 k_2 I^2}{Q}\,dT \tag{5}$$

since $Q_J = Q_2 - Q_1$. Equation (5) expresses the Carnot refrigerator power for arbitrary heat current distributions, $Q(T)$, in the binary lead. It should be noted that the first term of Equation (5) is integrated for the entire lead, while the second term is integrated only for the metal part since it accounts for the cooling of the generated heat.

The minimization of the refrigerator power, Equation (5), should be performed subject to the condition that the geometry of the HTS remains constant. The geometric constraint can be put into an integral form.

$$\frac{L_1}{A_1}=\int_{T_L}^{T_J}\frac{k_1}{Q}\cdot dT \tag{6}$$

where L and A are the length and the cross-sectional area of the lead, respectively. The variational problem of minimizing Equation (5) subject to Equation (6) is identical to minimizing a functional

$$F=\int_{T_L}^{T_H}\left\{T_H\frac{Q}{T^2}+\left(\frac{T_H}{T}-1\right)\frac{\rho_2 k_2 I^2}{Q}\,U(T-T_J)\right. \tag{7}$$

$$\left.-\lambda\frac{k_1}{Q}\left[1-U(T-T_J)\right]\right\}dT$$

without any constraints, where λ is the Lagrange multiplier and $U(T-T_J)$ is a unit step function whose value is 1 if $T \geq T_J$ and 0 otherwise.

Using the method of calculus of variations, the optimal heat current distribution to minimize F can be found as a function of temperature when the end and the joint temperatures are given.

$$Q_{opt}(T)=\begin{cases}\dfrac{A_1}{L_1}\sqrt{k_1}\cdot T\cdot\displaystyle\int_{T_L}^{T_J}\frac{\sqrt{k_1}}{T}\,dT & \text{for } T<T_J\\[2ex] I\cdot T\cdot\sqrt{\left(\dfrac{1}{T}-\dfrac{1}{T_H}\right)\rho_2 k_2} & \text{for } T\geq T_J\end{cases} \tag{8}$$

The optimal dimensions of the metal part to minimize F are given by the relation

$$\left(\frac{L_2}{A_2}\right)_{opt}=\frac{\sqrt{T_H}}{I}\int_{T_J}^{T_H}\sqrt{\frac{k_2}{\rho_2 T(T_H-T)}}\cdot dT \tag{9}$$

The optimal heat current distribution, Equation (8), is now substituted into Equation (5) and the minimum refrigerator power is finally obtained.

$$(W_{ref})_{min}=T_H\left[\frac{A_1}{L_1}\left(\int_{T_L}^{T_J}\frac{\sqrt{k_1}}{T}\,dT\right)^2\right. \tag{10}$$

$$\left.+2I\int_{T_J}^{T_H}\frac{1}{T}\sqrt{\left(\frac{1}{T}-\frac{1}{T_H}\right)\rho_2 k_2}\,dT\right]$$

Equation (10) represents the minimum refrigerator power in case the Carnot refrigerators are optimally distributed and the dimensions of the metal part are optimally designed. The first term is the minimum power required to cool the HTS part, which is proportional to the cross-sectional area and inversely proportional to the length, but is independent of the current since no heat has been assumed to be generated. The second term is the power required to cool the metal part, which is proportional to the current level. The mathematical exactness of Equation (10) can be confirmed by comparing its two limiting cases with the previously published works. When the joint temperature is equal to the cold end temperature, Equation (10) becomes

$$(W_{ref})_{min}=2I\cdot T_H\int_{T_L}^{T_H}\frac{1}{T}\sqrt{\left(\frac{1}{T}-\frac{1}{T_H}\right)\rho_2 k_2}\,dT \tag{11}$$

which is identical to the minimum power to refrigerate a metallic lead, as derived by Hilal[12]. If it is assumed that the joint temperature is equal to the warm end temperature and the HTS is still superconducting, Equation (10) becomes

$$(W_{ref})_{min}=T_H\frac{A_1}{L_1}\left(\int_{T_L}^{T_H}\frac{\sqrt{k_1}}{T}\,dT\right)^2 \tag{12}$$

which is also identical to the well-known result by Bejan and Smith[13], because the optimization of the 'room-temperature' superconductor leads would be the same as that of the mechanical supports for cryogenic apparatus from the thermodynamic point of view.

In published reports about current lead[1–4,8,12], the refrigerator power per unit operating current is more significant in the design and analysis. The minimum refrigerator power per unit current can be expressed as

$$\left(\frac{W_{ref}}{I}\right)_{min}=T_H\left[\frac{1}{J_1 L_1}\left(\int_{T_L}^{T_J}\frac{\sqrt{k_1}}{T}\,dT\right)^2\right. \tag{13}$$

$$+ 2\int_{T_J}^{T_H} \frac{1}{T}\sqrt{\left(\frac{1}{T} - \frac{1}{T_H}\right)\rho_2 k_2 \mathrm{d}T}\Bigg]$$

where J_1 is the current density at the HTS. It is very worth while noticing in Equation (13) that the minimum power per unit current is a function of J_1, L_1 and T_J only, when the material properties and the end temperatures are given.

The minimum refrigerator power per unit current has been calculated with Equation (13) for various values of the current density and the joint temperature in a copper + Bi2223 binary lead. *Figure 2* shows contours of the minimum power per unit current on a current density vs. joint temperature (J_1-T_J) diagram for L_1 = 20 cm. The thermal conductivity of Bi2223 is taken from Herrmann et al.[3] and the properties of copper are taken from Maehata et al.[14] for RRR = 60. The cold and warm end temperatures of the lead are fixed at 4 K and 300 K, respectively, throughout this paper. Generally speaking, the refrigerator power per unit current decreases as the current density or the joint temperature increases. However, when the joint temperature is low or the current density is high, the power per unit current is almost independent of the current density since the cooling load for the HTS or the first term in Equation (13) is relatively small. On the contrary, when the joint temperature is high or the current density is low, the first term is dominant and the current density is relatively more significant in the total power per unit current than the joint temperature.

Since Equation (13) has been derived with the assumption that the HTS does not generate heat, the superconductivity should be confirmed by incorporating the critical properties of the HTS, which will establish the final process of the optimization. The current density of Bi2223 can be represented reasonably well by a linear function of temperature[1,5],

$$J_C = J_{C0}\left(1 - \frac{T}{T_C}\right) \qquad (14)$$

where J_{C0} is the critical current density at 0 K and varies over a wide range, depending upon the size, the shape and the fabrication method as well as the applied magnetic field. For the purpose of more quantitative discussions in this paper, it is assumed that J_{C0} = 10 000 A/cm^2 and T_C = 104 K for Bi2223 at zero magnetic field[11] and Equation (14) is plotted on the J_1-T_J diagram of *Figure 2*. Clearly, there exist unique optimal values in the current density and the joint temperature, to minimize the refrigerator power per unit current while the HTS is superconducting, as indicated by the dot. At higher joint temperatures and smaller current densities than the optima, more refrigerator power per unit current is required because of greater refrigerator power to cool the HTS. At lower joint temperatures and larger current densities, more power per unit current is also required because of greater power to cool the metal part of the lead. The absolute minimum of the refrigerator power per unit current is about 0.0569 W/A, which is only 24% of the power per unit current required to refrigerate the optimized metal lead as given by Equation (11). The corresponding optimal values of J_1 and T_J are 481 A/cm^2 and 99 K, respectively. It should be remembered that this refrigerator power per unit current is the absolute minimum as a thermodynamic limit for the Cu + Bi2223 lead when every other design parameter has been optimized except the length and the critical current density for Bi2223.

The above procedure has been repeated for various values of L_1 and J_{C0} of Bi2223 and the results have been plotted in *Figure 3*. The absolute minimum refrigerator power per unit current decreases as L_1 or J_{C0} increases. However, the minimum power does not vary significantly if L_1 is greater than about 10 cm, which means that the length of the HTS does not need to be very long as well as it is optimally cooled. It is also noticed that the minimum power is not very sensitive to the critical current density of the HTS if J_{C0} is greater than about 10 000 A/cm^2. As L_1 is reduced to zero, the minimum power per unit current approaches 0.222 W/A regardless of the current density because it is the absolute minimum for the optimized copper lead as given by Equation (11). If L_1 or J_{C0} is infinitely large, the minimum power per unit current is asymptotically reduced to 0.0516 W/A, which can be directly calculated by the asymptotic behavior of Equation (13),

$$\left(\frac{W_{ref}}{I}\right)_{min} = 2T_H\int_{T_C}^{T_H} \frac{1}{T}\sqrt{\left(\frac{1}{T} - \frac{1}{T_H}\right)\rho_2 k_2 \mathrm{d}T} \qquad (15)$$

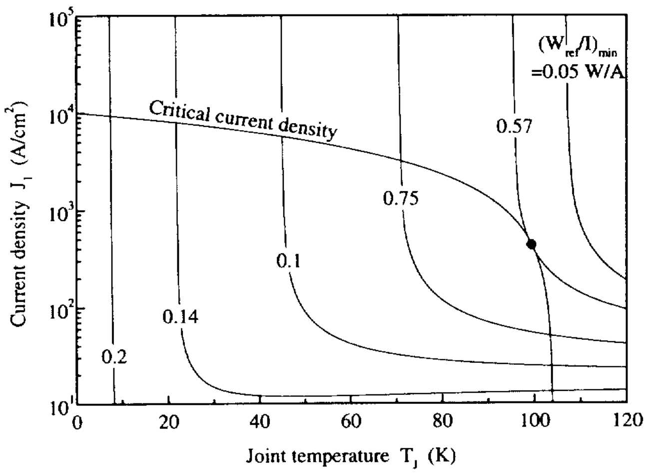

Figure 2 Contours of absolute minimum refrigerator power per unit current and critical current density on current density (J_1) vs. joint temperature diagram (T_J) for Cu + Bi2223 when L_1 = 20 cm and J_{C0} = 10 000 A/cm^2

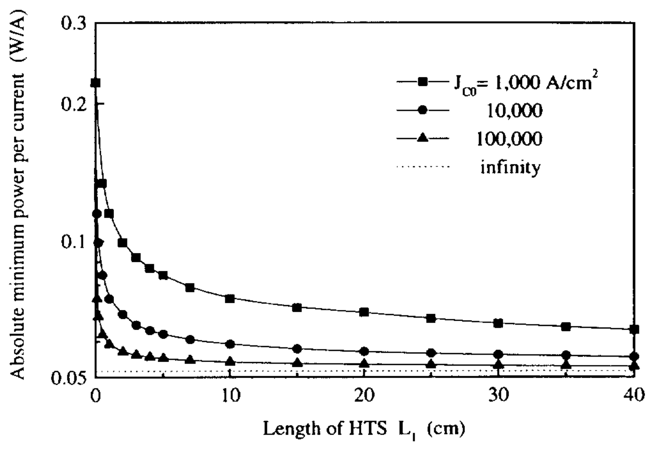

Figure 3 Absolute minimum refrigerator power per unit current as a function of the HTS (Bi2223) length for various values of critical current density

Table 1 Summary of the refrigerator power per unit current for the optimized Cu + Bi2223 current leads

Critical current density at 0 K (A/cm^2)	Absolute minimum with distributed refrigeration (W/A)			Minimum in 2-stage refrigeration (W/A)		
	$L_1 = 0$	$L_1 = 20$ cm	$L_1 = \infty$	$L_1 = 0$	$L_1 = 20$ cm	$L_1 = \infty$
1000		0.0695			0.122	
10 000	0.222	0.0569		3.10[a]	0.0907	
100 000		0.0534			0.0806	
∞	-		0.0516	-		0.0759

[a]Single-stage cooling at 4 K.

which is the simple case that no refrigerator power is required for the HTS and the cold end temperature of the optimized metal lead is the critical temperature of the HTS. Some of the meaningful values in the minimum refrigerator power per unit current for the Cu + Bi2223 binary lead are summarized in *Table 1*.

Optimization of two-stage refrigeration

The distributed Carnot refrigerators in the previous section were useful in finding the absolute minimum as a thermodynamic limit, but could not be realized in practice. In most practical cases for the conduction-cooled superconducting systems, a two-stage refrigerator should be employed[10,11]. The joint of the two parts is cooled by the first stage of the refrigerator and the cold end of the HTS lead is cooled by the second stage, as shown in *Figure 4*. The refrigerator power for the cooling can be generally expressed as,

$$W_{\mathrm{ref}} = \left(\frac{T_{\mathrm{H}}}{T_{\mathrm{L}}} - 1\right)\cdot\frac{Q_{\mathrm{L}}}{FOM_{\mathrm{L}}} + \left(\frac{T_{\mathrm{H}}}{T_{\mathrm{J}}} - 1\right)\cdot\frac{Q_{\mathrm{J}}}{FOM_{\mathrm{J}}} \tag{16}$$

where *FOM* is the figure of merit, defined as the ratio of actual coefficient of performance to Carnot coefficient of performance.

For the HTS part of the lead, the heat current is constant along the axis and is identical to the cooling load at the cold end, since no heat is generated.

$$Q_{\mathrm{L}} = Q_1 = \frac{A_1}{L_1}\int_{T_{\mathrm{L}}}^{T_J} k_1\cdot \mathrm{d}T \tag{17}$$

For the metal lead, the change of the heat current is due to the generated heat as shown in *Figure 4* or $\mathrm{d}Q = -\,\mathrm{d}Q_{\mathrm{gen}}$, which is substituted into Equation (3).

$$\mathrm{d}Q = -\frac{\rho_2 k_2 I^2}{Q}\,\mathrm{d}T \tag{18}$$

Equation (18) is multiplied by Q on both sides and integrated over the metal length.

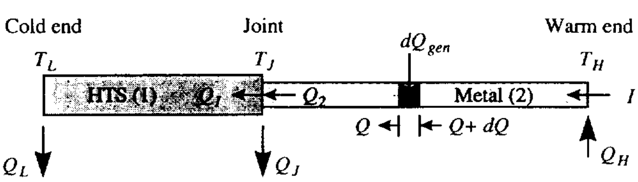

Figure 4 Binary current lead cooled at cold end and at joint by a two-stage refrigerator

$$\frac{1}{2}(Q_{\mathrm{H}}^2 - Q_2^2) = -I^2\int_{T_{\mathrm{J}}}^{T_H}\rho_2 k_2\cdot \mathrm{d}T \tag{19}$$

which can be rearranged for the heat current from the metal lead to the joint Q_2.

$$Q_2 = \sqrt{Q_{\mathrm{H}}^2 + 2I^2\int_{T_{\mathrm{J}}}^{T_H}\rho_2 k_2\cdot \mathrm{d}T} \tag{20}$$

It is immediately observed that Q_2 has its minimum when the heat current at the warm end, Q_{H}, is zero. If Q_{H} has a positive value, Q_2 is larger than the minimum because of the excessive heat conduction through the metal lead. If Q_{H} has a negative value on the contrary, Q_2 is also larger because of the excessive heat generation. The axial temperature gradient should be zero at the warm end when the heat conduction and the heat generation are optimally balanced. This condition for the minimum is identical to the case of the conventional vapor-cooled metal lead[15].

The minimum heat current to the joint is now found in a closed form by letting $Q_{\mathrm{H}} = 0$.

$$(Q_2)_{\min} = I\sqrt{2\int_{T_{\mathrm{J}}}^{T_H}\rho_2 k_2\cdot \mathrm{d}T} \tag{21}$$

At any arbitrary axial location of the metal lead, the optimal heat current can be obtained as a function of temperature by integrating Equation (18) from the point to the warm end and letting $Q_{\mathrm{H}} = 0$.

$$Q_{\mathrm{opt}}(T) = I\sqrt{2\int_{T}^{T_H}\rho_2(\tau)k_2(\tau)\cdot \mathrm{d}\tau} \tag{22}$$

from which the optimal dimensions of metal are found.

$$\left(\frac{L_2}{A_2}\right)_{\mathrm{opt}} = \frac{1}{\sqrt{2}\cdot I}\int_{T_{\mathrm{J}}}^{T_H}\frac{k_2}{\sqrt{\int_{T}^{T_H}\rho_2(\tau)k_2(\tau)\cdot \mathrm{d}\tau}}\cdot \mathrm{d}T \tag{23}$$

Equations (21) and (23) are very simple and useful expressions for the conduction-cooled metal lead, but have never been reported as far as the present authors are aware. The similar optimal conditions were reported in previous publications with simple assumptions for the material properties, which could be derived as special cases of these general expressions. If the electrical resistivity and the thermal conductivity are assumed to be constant, Equations (21) and (23) are reduced to

$$(Q_2)_{\min} = I\sqrt{2\rho_2 k_2 (T_H - T_J)} \tag{24}$$

and

$$\left(\frac{L_2}{A_2}\right)_{\text{opt}} = \frac{1}{I}\sqrt{\frac{2k_2}{\rho_2}(T_H - T_J)} \tag{25}$$

respectively, which were described by Seol and Hull[7]. For materials that obey the Wiedemann–Franz law, $\rho \cdot k = L_0 T$, Equations (21) and (23) are directly integrated.

$$(Q_2)_{\min} = I\sqrt{L_0(T_H^2 - T_J^2)} \tag{26}$$

and

$$\left(\frac{L_2}{A_2}\right)_{\text{opt}} = \frac{1}{I}\int_{T_J}^{T_H} \frac{k_2}{\sqrt{L_0(T_H^2 - T^2)}}\cdot \mathrm{d}T \tag{27}$$

as Yang and Pfotenhauer[8] mentioned. The minimum cooling load at joint is obtained from Equations (17) and (21).

$$(Q_J)_{\min} = I\sqrt{2\int_{T_J}^{T_H} \rho_2 k_2 \cdot dT} - \frac{A_1}{L_1}\int_{T_L}^{T_J} k_1 \cdot \mathrm{d}T \tag{28}$$

The refrigerator power required for the two-stage cooling is associated with the performance characteristics of the refrigerator, represented by the *FOM*'s in Equation (16). The optimization of the lead cooled by a two-stage Carnot or reversible refrigerator is considered first and some aspects of the optimization of the lead cooled by actual refrigerators are discussed later.

The minimum refrigerator power with a two-stage Carnot refrigerator is derived by substituting Equations (17) and (21) into Equation (16) and setting the two *FOM*'s to unity.

$$(W_{\text{ref}})_{\min} = T_H\left[\left(\frac{1}{T_L} - \frac{1}{T_J}\right)\frac{A_1}{L_1}\int_{T_L}^{T_J} k_1 \cdot \mathrm{d}T + \left(\frac{1}{T_J} - \frac{1}{T_H}\right)\cdot I \cdot \sqrt{2\int_{T_J}^{T_H} \rho_2 k_2 \cdot \mathrm{d}T}\right] \tag{29}$$

Equation (29) is divided by the current and the minimum refrigerator power per unit current is finally found.

$$\left(\frac{W_{\text{ref}}}{I}\right)_{\min} = T_H\left[\left(\frac{1}{T_L} - \frac{1}{T_J}\right)\frac{1}{J_1 L_1}\int_{T_L}^{T_J} k_1 \cdot \mathrm{d}T + \left(\frac{1}{T_J} - \frac{1}{T_H}\right)\sqrt{2\int_{T_J}^{T_H} \rho_2 k_2 \cdot \mathrm{d}T}\right] \tag{30}$$

which is a function of J_1, L_1 and T_J only, as in Equation (13). The minimum in Equation (30) has been obtained by optimizing the dimensions of the metal lead and assuming a reversible refrigerator.

In the same manner as in the previous section, the minimum refrigerator power per unit current has been calculated using Equation (30) and contours of the constant power per unit current have been plotted on a J_1-T_J diagram as shown in *Figure 5*, for a copper + Bi2223 binary lead with L_1 equal to 20 cm. The contour curves have basically the same shapes as in *Figure 2*, even though Equation (30) may be completely different from Equation (13). The physical meanings for the shapes of the curves are also the same as in *Figure 2*. However, it can be observed that the minimum power per unit current in *Figure 5* increases more sharply as the joint temperature decreases and that the curves at low J_1 region rise up again on J_1-T_J diagram as the joint temperature increases. These two behaviors are obviously due to more power required in the two-stage refrigeration than in the distributed refrigeration, particularly at very low temperatures. The concave shape of the curves at low J_1 region indicates that for a constant current density, the optimum joint temperature to minimize the power per unit current should be significantly lower than the critical temperature, as discussed by Yang and Pfotenhauer[8].

The critical current density as a function of temperature has been plotted in *Figure 5*, again with the assumption that J_{C0} = 10 000 A/cm² and T_C = 104 K for Bi2223[11]. It is also clear in the two-stage refrigeration that there exist unique optimal values in the current density and the joint temperature, to minimize the refrigerator power per unit current while the HTS is superconducting, as indicated by the dot. When J_1 = 577 A/cm² and T_J = 98 K, the power per unit current in two-stage refrigeration has a theoretical minimum of 0.0907 W/A, which is approximately 59% larger than the absolute minimum with the distributed refrigeration for the same length of the HTS.

The above procedure has been repeated for various

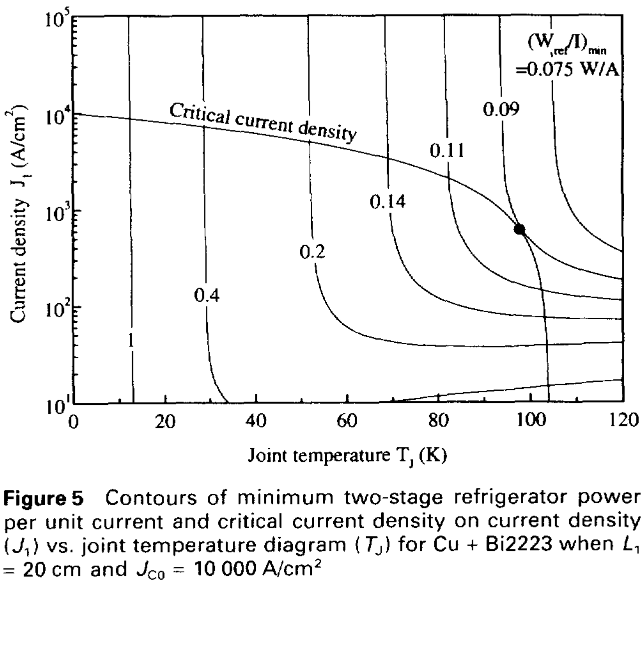

Figure 5 Contours of minimum two-stage refrigerator power per unit current and critical current density on current density (J_1) vs. joint temperature diagram (T_J) for Cu + Bi2223 when L_1 = 20 cm and J_{C0} = 10 000 A/cm²

values of L_1 and J_{C0} of Bi2223 and the results have been plotted in *Figure 6*. It is observed again in two-stage refrigeration that the minimum refrigerator power per unit current does not decrease significantly as L_1 increases beyond about 10 cm or J_{C0} increases beyond about 10 000 A/cm^2. As L_1 decreases to zero, the minimum power per unit current approaches to 3.10 W/A, which is the minimum power for the single-staged cooling of the optimized copper lead as given by

$$\left(\frac{W_{\rm ref}}{I}\right)_{\min} = \left(\frac{T_{\rm H}}{T_{\rm L}} - 1\right)\sqrt{2\int_{T_{\rm L}}^{T_H} \rho_2 k_2 \cdot {\rm d}T} \qquad (31)$$

If L_1 or J_{C0} is infinitely large, the minimum power per unit current is asymptotically reduced to 0.0759 W/A, which can be directly calculated by the asymptotic behavior of Equation (30),

$$\left(\frac{W_{\rm ref}}{I}\right)_{\min} = \left(\frac{T_{\rm H}}{T_C} - 1\right)\sqrt{2\int_{T_C}^{T_H} \rho_2 k_2 \cdot {\rm d}T} \qquad (32)$$

which is the simple case that the refrigerator power to cool the cold end is negligible and the joint temperature is the critical temperature of the HTS. It can be also shown that both of the optimal values for J_1 and T_J decrease as L_1 increases. The minimum power per unit current for the two-stage refrigeration is summarized and compared with the absolute minimum in *Table 1*.

The preceding optimization for the two-stage refrigeration of the binary lead is a theoretical limit, because the refrigerator has been assumed to be reversible and the HTS is marginally superconducting at the optimal conditions as shown in *Figure 5*. It is beyond the scope of this paper to include the detailed characteristics of actual two-stage refrigerators, the detailed critical properties for various HTS materials or the thermal stability of the conduction-cooled HTS. On the other hand, that those three essential aspects should be considered in the practical design are shortly discussed here in order to highlight the usefulness of the presented optimization method.

If the *FOM*'s are known for the actual two-stage refrigerator, the minimum refrigerator power per unit current for the optimized dimensions of the metal lead can be derived from Equations (16), (17) and (21).

$$\left(\frac{W_{\rm ref}}{I}\right)_{\min} = \frac{1}{FOM_{\rm L}}\left(\frac{T_{\rm H}}{T_{\rm L}} - 1\right)\frac{1}{J_1 L_1}\int_{T_{\rm L}}^{T_J} k_1 \cdot {\rm d}T \qquad (33)$$

$$+ \frac{1}{FOM_{\rm J}}\left(\frac{T_{\rm H}}{T_{\rm J}} - 1\right)\left(\sqrt{2\int_{T_{\rm J}}^{T_H} \rho_2 k_2 \cdot {\rm d}T} - \frac{1}{J_1 L_1}\int_{T_{\rm L}}^{T_J} k_1 \cdot {\rm d}T\right)$$

Generally, the *FOM* of a cryogenic refrigerator depends on the cold head temperature, the type of the refrigeration cycle and the capacity of the refrigerator. The typical values can be found in a range between 1/10 and 1/30 for most of the refrigerators that are available for the cooling of the binary current leads[2,8,9,13]. With these values and Equation (33), the actual refrigerator power per unit current can be calculated and plotted on a J_1-T_J diagram in the same way as in *Figure 5*. The magnitudes of the power per unit current would be greater by 10 to 30 times than those in *Figure 5* and the shape of the contour curves would be more concave at a low J_1 region if $FOM_{\rm L}$ is smaller than $FOM_{\rm J}$.

The critical properties of the HTS are strongly dependent on the size, the shape and the fabrication method as mentioned above. Once the critical current density is given as a function of temperature for a specific HTS lead and under a specific magnetic field, it is plotted on the J_1-T_J diagram together with the contours of the actual refrigerator power per unit current so that the optimal point may be determined to minimize the power per unit current. Because of the shape of the contours, a smaller critical current density would result in a greater refrigerator power and a lower optimum joint temperature. The reader is reminded that the critical current density is involved in this step of the optimization separately from the thermophysical or the electrical property.

Because of the nature of the superconductivity, the theoretical optimum to minimize the refrigerator power per unit current is always determined at a marginally superconducting state as shown in *Figure 5*. In practice, the HTS current leads should be designed such that the current density and the joint temperature are lower than their theoretical optima, in order to be stable for any thermal disturbances. The presented optimization method could be very useful also in finding the design point of the practical cases. On the J_1-T_J diagram, the best design points should be maintained in a certain distance from the critical line and still have a minimum refrigerator power per unit current. The points could be located on a curve which intersects the theoretical optimum point and is perpendicular to the contour curves.

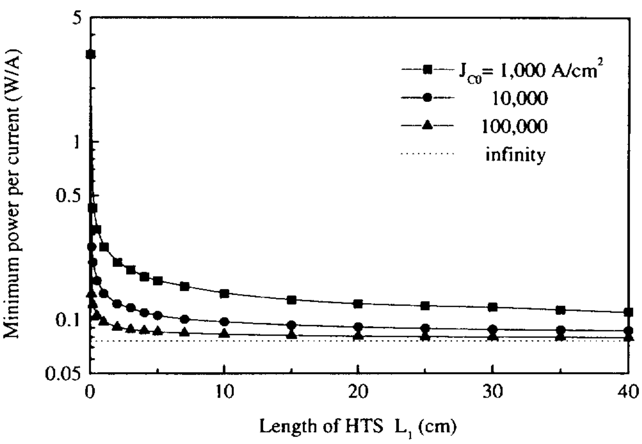

Figure 6 Minimum two-stage refrigerator power per unit current as a function of the HTS (Bi2223) length for various values of critical current density

Conclusion

A theoretical optimization has been performed for conduction-cooled binary (metal + HTS) current leads. In the first part of the optimization, it has been successfully shown by analytical and graphical methods that the refrigerator power per unit current has an absolute minimum as thermodynamic limit. The absolute minimum is obtained for a given length of the HTS lead, when the lead is cooled by optimally distributed refrigerators along its length and the current density of the HTS has its critical value at an optimal temperature for the joint of the two parts. In the second part, the two-stage cooling of the current lead has been optimized by similar analytical and graphical methods. A theoretical minimum power per unit current for the two-stage refrigeration is also found for a given length of the HTS lead, when the dimensions of the metal lead are optimized and the current density of the HTS has its critical value at an optimal temperature for the joint. Some general aspects in practical design are also discussed in association with the characteristics of actual refrigerators and the critical properties of the HTS.

Acknowledgements

This paper was accomplished under the Financial Support Program for Faculty Research Abroad provided by Korea Research Foundation. The National High Magnetic Field Laboratory is jointly supported by the State of Florida and the National Science Foundation under grant DHR-9527035.

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