AN EXACT EXPRESSION FOR SHUTTLE HEAT TRANSFER

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ABSTRACT

A new and general analytical expression is obtained for the shuttle heat transfer rate in displacer-cylinder systems. The shuttle heat transfer occurs when the displacer has a reciprocating motion over the cylinder with an axial temperature gradient. The heat transfer might be important in small cryogenic refrigerators because it represents a loss of refrigeration in addition to the wall conduction. The oscillating temperature distributions of both the displacer wall and the cylinder wall are exactly obtained as functions of time and space from the conduction equations by introducing the complex temperatures. From the cyclic steady state solution of the temperatures, a simple mathematical expression is derived to calculate the shuttle heat transfer. The expression includes the axial temperature gradient, the stroke and the angular speed of the reciprocating motion, the heat transfer coefficient between two walls, and the thermal properties (such as thermal conductivity, density, or specific heat) of two walls. The usefulness of the results is justified by the approximate solutions of previous works and the physical interpretations are presented.

INTRODUCTION

Many cryogenic refrigerators have displacers and cylinders as their components. When there exists an axial temperature gradient along the walls of the displacer and the cylinder, the reciprocating motion of the displacer causes an extra heat transfer from the high to the low temperature region in addition to the wall conduction. This motional heat transfer has been called shuttle heat transfer. The shuttle heat transfer could be important in Stirling, Gifford-McMahon, Vuilleumier or Claude refrigerators because it represents a loss of refrigeration.

The basic principle of the shuttle heat transfer can be explained in Figure 1. While the displacer is moved toward the high temperature side from the center of the stroke, the displacer wall has a lower temperature than the cylinder wall of the same axial position. Therefore heat is transferred from the cylinder to the displacer. Similarly, while the displacer is moved toward the low temperature side, the displacer wall has a higher temperature and heat is transferred from the displacer to the cylinder. Because of the heat capacity of the displacer wall, the oscillations of the displacer temperature and of the displacer movement are not in phase. At a fixed axial position, the enthalpy flow accompanying the displacer motion to the low temperature direction is always greater than that to the high temperature direction. The net enthalpy flow rate from the high to the low temperature region is the shuttle heat transfer.

Several mathematical expressions have been introduced up to date to estimate the amount of the shuttle heat transfer. Zimmerman and Longthworth[1] derived an expression based upon the assumptions that the walls of the displacer and the cylinder have an infinite heat capacity and that the displacer has a square-wave motion. Rios[2] presented an approximate solution by linearization and application of Fourier series. Radebaugh and Zimmerman[3] developed an approximate solution by heat conduction analysis for a semi-infinite plate with a sinusoidal surface temperature distribution. In their analysis, the thermal resistance of gas in the gap between the walls was neglected in the calculation of the wall temperatures, and was included later by an approximate method. Martini[4] summarized these expressions to estimate the shuttle heat transfer. Recently, Nishio et al.[5] presented an approximate solution including the temperature oscillations for both of the displacer and the cylinder. In their work, a thermal resistance model was employed and the results were compared with those of their own numerical simulation.

Recently the authors published a new mathematical solution for the shuttle heat transfer, including the wall properties of both the displacer and the cylinder[6]. In this paper, a simpler and more useful form of the expression is presented and the physical interpretations are discussed.

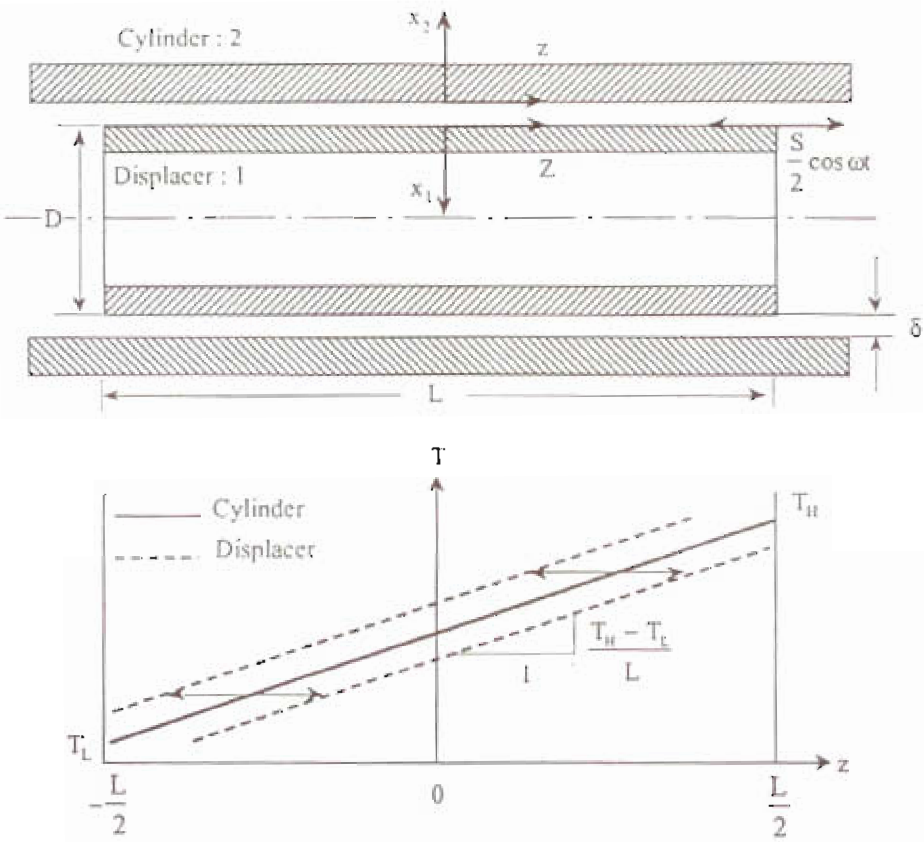

Figure 1. Schematic representation of geometry and temperature distribution for displacer-cylinder system.

FORMULATION AND SOLUTION

Analysis Model

As shown in Figure 1, both the displacer and the cylinder have the same axial linear temperature gradient, $(T_H-T_L)/L$, and the displacer reciprocates with angular speed ω and stroke S. The stationary axial coordinate z is measured from the center of the cylinder and the moving axial coordinate Z is measured from the center of the displacer. The radial coordinates x_1 and x_2 are measured from the facing surfaces of the displacer and the cylinder, respectively. The subscripts 1 and 2 denote the dispalcer and the cylinder, respectively.

For simplicity, the system is assumed be in a cyclic steady state with a sinusoidal motion with angular velocity ω and stoke S. The two axial coordinates are related with time, t, by the following relation.

$$z = Z + \frac{S}{2}\cos\omega t \tag{1}$$

The wall thicknesses of the displacer and the cylinder are assumed to be much smaller than the diameters, but greater than the thermal penetration depths so that the displacer and the cylinder are considered as two semi-infinite flat plates facing each other. These assumptions can be justified in terms of the real situation where the shuttle heat transfer is significant as checked later. It should be noted that the heat capacities of the displacer and the cylinder are not assumed to be infinity and that the temperature of the walls is a function of the axial position (z or Z), the radial position (x), and time (t).

The governing equations for the temperature of the cylinder, T_2, and the temperature of the displacer, T_1, are

$$\frac{1}{\alpha_2}\frac{\partial T_2(t,x_2,z)}{\partial t} = \frac{\partial^2 T_2(t,x_2,z)}{\partial {x_2}^2} \tag{2}$$

$$\frac{1}{\alpha_1}\frac{\partial T_1(t,x_1,Z)}{\partial t} = \frac{\partial^2 T_1(t,x_1,Z)}{\partial {x_1}^2} \tag{3}$$

respectively. It is noted that the axial coordinate, z, of Eq. (2) is stationary while the axial coordinate, Z, of Eq. (3) is moving with the displacer and that α's are the thermal diffusivities.

Two boundary conditions are

$$T_2(t,x_2=\infty,z) = T_0 + \frac{T_H - T_L}{L}z \tag{4}$$

$$T_1(t,x_1=\infty,Z) = T_0 + \frac{T_H - T_L}{L}Z \tag{5}$$

where T_0 is the cylinder temperature at $z=0$, and radially far from the surface. It is clear that two temperatures of Eq. (4) and (5) are same when $z=Z$ or the displacer is at the center of the stroke. Two other boundary conditions are obtained from the heat transfer relation between the facing surfaces of the displacer and the cylinder.

$$-k_1 \frac{\partial T_1\left(t, 0, z-\frac{S}{2}\cos\omega t\right)}{\partial x_1} = h\left\{T_2(t,0,z) - T_1\left(t,0,z-\frac{S}{2}\cos\omega t\right)\right\} = k_2 \frac{\partial T_2(t,0,z)}{\partial x_2} \qquad (6)$$

where k's are the thermal conductivities of the walls and h, is the heat transfer coefficient between the two surfaces. In Eq. (6), the axial position of displacer, Z, was replaced with a function of z and t via Eq. (1).

Wall Temperatures

The wall temperatures for the displacer and the cylinder are obtained by solving Eq. (2) and (3) simultaneously with the boundary conditions, Eq. (4), (5), and (6), by introducing complex temperatures[6]. For the displacer wall,

$$\begin{aligned} T_1(t, x_1, z) &= T_0 + \frac{T_H - T_L}{L}\left[z - \frac{S}{2}\cos\omega t\right] \\ &+ \frac{\frac{T_H - T_L}{L}\frac{S}{\sqrt{2}}b_1}{\sqrt{(b_1+b_2)^2 + (b_1+b_2+1)^2}} exp\left[-\sqrt{\frac{\omega}{2\alpha_1}}x_1\right] cos\left[\omega t - \sqrt{\frac{\omega}{2\alpha_1}}x_1 + \frac{\pi}{4} - \phi\right] \end{aligned} \qquad (7)$$

where

$$tan\phi = \frac{b_1 + b_2 + 1}{b_1 + b_2} \qquad (8)$$

$$b_1 \equiv \frac{h}{k_1}\sqrt{\frac{\alpha_1}{2\omega}} \qquad (9)$$

$$b_2 \equiv \frac{h}{k_2}\sqrt{\frac{\alpha_2}{2\omega}} \qquad (10)$$

The dimensionless numbers, b_1 and b_2, are Biot numbers whose characteristic length is a thermal penetration depth in the walls.

Shuttle Heat Transfer Rate

With the obtained temperature of the displacer wall, the shuttle heat transfer can now be calculated. At an arbitrary axial position, the shuttle heat transfer can be defined as the net enthalpy flow rate from the high to the low temperature side, or negative z direction. A cycle-averaged net enthalpy flow rate of the displacer wall can be obtained by integrating over a period and the displacer wall area.

$$Q = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^{\infty} \rho_1 c_1 T_1(t, x, z)\cdot\left(\frac{\omega S}{2}sin\,\omega t\right)\pi D dx \cdot dt \qquad (11)$$

where the parentheses denotes the velocity of displacer in the negative z direction and ρ and c denote density and specific heat of the walls, respectively. The result of integrating Eq. (11) is the new expression for shuttle heat transfer.

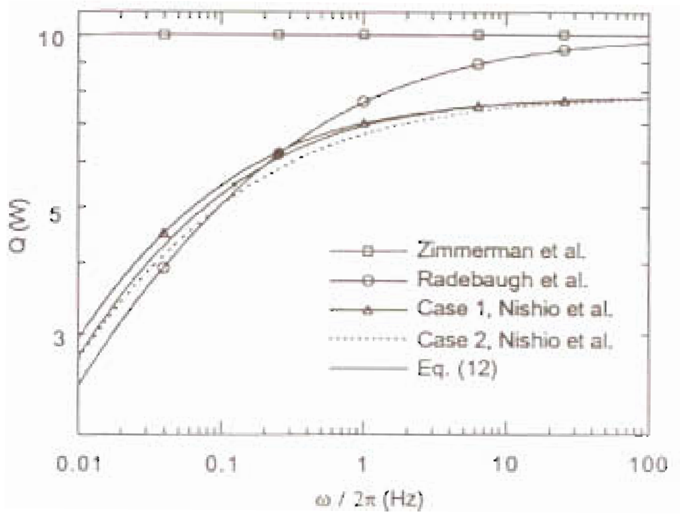

Figure 2. Comparison of current and existing expressions for shuttle heat transfer rate as a function of frequency.

$$\dot{Q} = \frac{T_H - T_L}{L}\frac{\pi D S^2 h}{8}\frac{b_1 + b_2 + 1}{(b_1 + b_2)^2 + (b_1 + b_2 + 1)^2} \tag{12}$$

It can be noted that the shuttle heat transfer rate is independent of z .

RESULTS AND DISCUSSION

The new expression, Eq. (12), could be compared with the existing data for a specific situation. The materials and the physical dimensions for this calculation are given in Table 1. The shuttle heat transfer rates were calculated for a variable frequency of oscillation by using the existing methods and the new method. In all cases, the heat transfer coefficient is obtained by

$$h = \frac{k_g}{\delta} \tag{13}$$

where k_g and δ are the thermal conductivity of gas and the gap between the walls, respectively.

Figure 2 shows the shuttle heat transfer as a function of the frequency of oscillation for four different cases. The case 1 and the case 2 of Nishio et al. indicate the results without and with temperature oscillation of the cylinder, respectively. The shuttle heat transfer from Zimmerman et al. is constant and much over-estimated due to the infinite heat capacity of the two walls. The result from Radebaugh et al. has greater values of shuttle heat transfer at high frequencies and lower values at low frequencies . The result from the case 1 of Nishio et al. has greater values than those from the case 2 and from Eq. (12). The cause for the difference is very obvious; the phase shift from the temperature oscillation of the displacer to that of the cylinder was over-estimated due to the heat capacity of the cylinder being infinite.

A fairly good agreement is observed between the case 2 of Nishio et al. and Eq. (12). The slight difference is due to the assumption of Nishio et al. that the phase shift of the surface temperature oscillation is neglected. Clearly, Eq. (12) is much more useful than

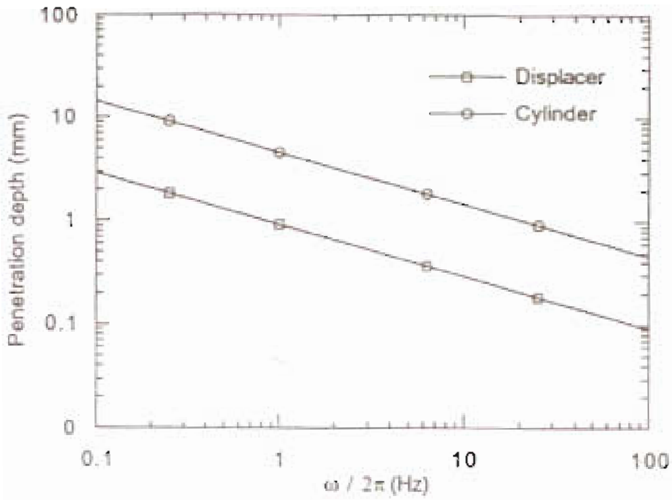

Figure 3. Thermal penetration depths of displacer and cylinder as a function of frequency.

Nishio et al., because the former is an exact solution that satisfies the governing equations and the boundary conditions while the latter is an approximate solution with several simplifying assumptions. In addition to the accuracy, it is also obvious that Eq. (12) is simpler to use.

As mentioned earlier, Eq. (12) is based on the model of two semi-infinite bodies. The validity of the model should be checked by seeing whether the thermal penetration depth is smaller than the thickness of the wall. Figure 3 shows the thermal penetration depth as a function of frequency. In the calculation, the same dimensions given in Table 1 were used, and the thermal penetration depth is defined as the distance from the surface to the point where the amplitude of the temperature oscillation is 1% of the amplitude of the surface temperature oscillation. As can be seen in Eq. (7), the penetration depth is proportional to the square root of α / ω. In this specific problem, the displacer penetration depth is always smaller than the wall thickness for all frequencies. On the other hand, the cylinder penetration depth is smaller than the wall thickness only when the frequency is greater than about 1.2 Hz in the specific situation.

Figure 4 shows the effect of the frequency and the gap between the walls on the shuttle heat transfer. Generally, the shuttle heat transfer increases as the frequency increases or as the gap decreases. It is observed that at low frequencies the heat transfer is strongly dependent on the frequency but is not much dependent on the gap. It is also observed that at high frequencies the heat transfer is strongly dependent on the gap but is not much dependent on the frequency. Quantitatively, the first observation is explained by an asymptotic expression of Eq. (12) for small ω or for small δ.

Table 1. Specifications for sample calculation

Displacer (Bakelite)	Diameter, D	125 mm
	Stroke, S	25 mm
	Thickness	3 mm
Cylinder (Stainless Steel)	Thickness	4 mm
	Temperature gradient	1 K / mm
Gas (Helium)	Temperature	160 K
	Gap, δ	0.4 mm

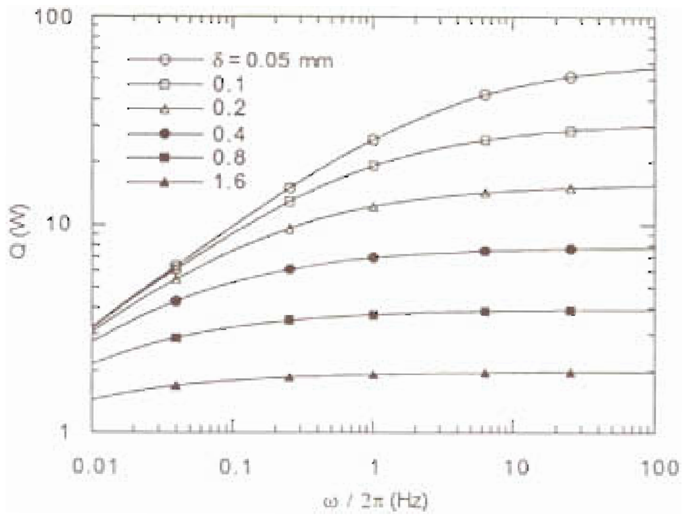

Figure 4. Shuttle heat transfer rate as a function of frequency for various gaps between the walls.

$$Q = \frac{T_H - T_L}{L} \frac{\pi D S^2}{8\sqrt{2}} \frac{\sqrt{\omega}}{\frac{1}{\sqrt{k_1 \rho_1 c_1}} + \frac{1}{\sqrt{k_2 \rho_2 c_2}}} \tag{14}$$

In words, the shuttle heat transfer is proportional to the square root of the frequency and independent of the gap when the Biot numbers defined by Eq. (9) and (10) are much greater than 1 or $b_1 + b_2 >> 1$. Eq. (14) is useful because it shows explicitly the effects of the wall properties on the heat transfer. Similarly, the second observation is explained by an asymptotic expression of Eq. (12) for large ω or for large δ.

$$Q = \frac{T_H - T_L}{L} \frac{\pi D S^2}{8} \frac{k_g}{\delta} \tag{15}$$

The shuttle heat transfer is independent of the frequency and inversely proportional to the gap when the Biot numbers are much smaller than 1 or $b_1 + b_2 << 1$. It can be noted that Eq. (15) is the same form of expression by Zimmerman and Longthworth[1] except the constant, because the thermal capacities of the two walls were assumed to be infinite in this extreme case.

CONCLUSIONS

A new analytical expression was presented for the shuttle heat transfer rate, including the wall properties of both the displacer and the cylinder. The exactness of the new expression was verified. The effects of the oscillating frequency, the gap between the walls, and the wall properties on the amount of heat transfer were discussed. Most of the phenomena were qualitatively mentioned in the previous works but could be quantitatively and precisely described via the expression. The results are expected to be directly applicable to many cryogenic systems where the shuttle heat transfer is significant.

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