An exact solution for shuttle heat transfer

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A new analytical solution has been obtained for the shuttle heat transfer rate in displacer/cylinder systems having an axial temperature gradient. The temperature oscillations of both the displacer wall and the cylinder wall were considered in the analysis. After the cyclic steady state solution was obtained for the temperatures of the walls by introducing complex temperatures, a simple mathematical expression was derived to calculate the shuttle heat transfer. The usefulness of the results was justified by approximate solution of previous work.

Keywords: shuttle heat transfer; displacer; complex temperatures

Nomenclature

c	Specific heat
D	Outer diameter of displacer
h	Heat transfer coefficient
k	Thermal conductivity
L	Length of displacer
Q	Shuttle heat transfer rate
S	Stroke of displacer
t	Time
T	Temperature
x	Radial distance from surface
Z	Axial co-ordinate in moving reference frame
z	Axial co-ordinate in stationary reference frame

Greek letters

α	Thermal diffusivity
δ	Gap between displacer and cylinder
ϕ	Phase angle
Γ	Axial temperature gradient
θ	Complex temperature
ρ	Density
ω	Angular speed
ψ	Amplitude of oscillation of surface temperature

Subscripts

1	Displacer
2	Cylinder
g	Gas in space between displacer and cylinder

Many cryogenic refrigerators have displacers/cylinders as their components. When an axial temperature gradient exists along the walls of the displacer and the cylinder, the reciprocating motion of the displacer causes extra heat transfer from the high to the low temperature region in addition to wall conduction. This motional heat transfer has been called shuttle heat transfer. The shuttle heat transfer could be important in Stirling, Gifford–McMahon, Vuilleumier and Claude refrigerators because it represents a loss of refrigeration.

The basic principle of shuttle heat transfer is explained in *Figure 1*. When the displacer is moved towards the high temperature side from the centre of the stroke, the displacer wall has a lower temperature than the cylinder wall with the same axial position. Therefore heat is transferred from the cylinder to the displacer. Similarly, when the displacer is moved towards the low temperature side, the displacer wall has a higher temperature and heat is transferred from the displacer to the cylinder. Because of the heat capacity of the displacer wall, the oscillations of displacer temperature and displacer movement are not in phase. At a fixed axial position, the enthalpy flow accompanying displacer motion in the low temperature direction is always greater than that in the high temperature direction. The net enthalpy flow rate from the high to the low temperature region is the shuttle heat transfer.

Several mathematical expressions have been introduced to date to estimate the size of the shuttle heat transfer. Zimmerman and Longthworth[1] derived an expression based on the assumptions that the walls of the displacer and the cylinder have an infinite heat capacity and that the displacer has a square-wave motion. Rios[2] presented an approximate solution by linearization and application of Fourier series. Radebaugh and Zimmerman[3] developed an approximate solution by heat conduction analysis for a semi-infinite plate with a sinusoidal surface temperature distribution. In

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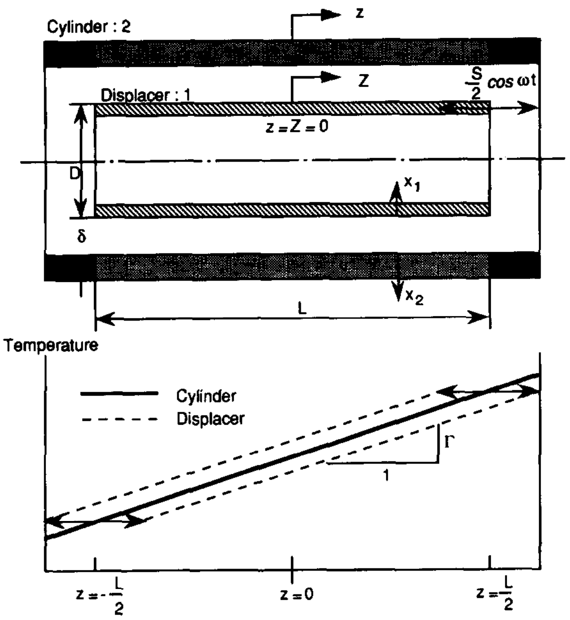

Figure 1 Schematic representation of shuttle heat transfer system and temperature distribution

their analysis, the thermal resistance of gas in the gap between the walls was neglected in the calculation of the wall temperatures and was included later by an approximate method. Martini[4] summarized these expressions to estimate the shuttle heat transfer. Recently, Nishio *et al.*[5] presented an approximate solution including the temperature oscillations for both the displacer and the cylinder. In this work, a thermal resistance model was employed and the results were compared with those from their own numerical simulation. To the present authors' knowledge an exact expression for the shuttle heat transfer has not been given in which the effects of temperature oscillation for both the displacer and the cylinder are taken into account.

This study presents a new mathematical solution for shuttle heat transfer, including the wall properties of both the displacer and the cylinder. The solution is mathematically exact, simple, and useful in practice. To justify the usefulness of the solution, the results are compared with other available information.

Analysis model

As shown in *Figure 1*, both the displacer and the cylinder have the same axial temperature gradient G, and the displacer reciprocates with angular speed w and stroke S. The stationary axial co-ordinate z is measured from the centre of the cylinder, and the moving axial co-ordinate Z is measured from the centre of the displacer. The radial co-ordinates x_1 and x_2 are measured from the facing surfaces of the displacer and cylinder, respectively.

For simplicity, the following assumptions are made in this analysis:

1 The system is in a cyclic steady state.
2 The reciprocating motion of the displacer is sinusoidal. Therefore the two axial co-ordinates have the following relation

$$z = Z + \frac{S}{2}\cos \omega t \tag{1}$$

3 The axial temperature gradient of the displacer and the cylinder is constant. This is true when the material properties are constant.
4 The heat transfer coefficient between the displacer and the cylinder walls is constant. This is approximately true when the end effect and the pressure oscillation of the gas are neglected.
5 The thermal penetration depths of the displacer and the cylinder are smaller than the wall thicknesses.
6 The wall thicknesses of the displacer and the cylinder are much smaller than the diameters, so that the displacer and the cylinder are considered as flat plates.
7 Radiation heat transfer between the displacer and the cylinder is negligible.

Most of these assumptions can be justified in terms of the real situation where the shuttle heat transfer is significant. Assumption (5), however, should be checked after the temperature distribution has been obtained. It should be noted that the heat capacities of the displacer and the cylinder are not assumed to be infinity and that the temperature of the walls is a function of the axial position, the radial position and time.

The governing equation for the temperature of the cylinder wall is the following heat conduction equation

$$\frac{1}{\alpha_2}\frac{\partial T_2(t,x_2,z)}{\partial t} = \frac{\partial^2 T_2(t,x_2,z)}{\partial x_2^2} \tag{2}$$

where the axial conduction term is neglected because of assumption (3). Similarly, the governing equation for the temperature of the displacer wall can be written as

$$\frac{1}{\alpha_1}\frac{\partial T_1(t,x_1,Z)}{\partial t} = \frac{\partial^2 T_1(t,x_1,Z)}{\partial x_1^2} \tag{3}$$

It is noted that the axial co-ordinate z of Equation (2) is stationary, while the axial co-ordinate Z of Equation (3) is moving with the displacer.

Two boundary conditions, according to assumptions (1) and (5), are

$$T_2(t,x_2 = \infty,z) = T_0 + \Gamma z \tag{4}$$

$$T_1(t,x_1 = \infty,Z) = T_0 + \Gamma Z \tag{5}$$

where T_0 is the cylinder temperature at $z = 0$ and radially far from the surface. It is clear that the two temperatures of Equations (4) and (5) are the same when $z = Z$ or the displacer is at the centre of the stroke. Two other boundary conditions are obtained from the heat transfer relation between the facing surfaces of the displacer and the cylinder

$$\begin{aligned} -k_1 \frac{\partial T_1\left(t,0,z - \frac{S}{2}\cos\omega t\right)}{\partial x_1} &= h\left\{T_2(t,0,z) - T_1\left(t,0,z - \frac{S}{2}\cos\omega t\right)\right\} \\ &= k_2 \frac{\partial T_2(t,0,z)}{\partial x_2} \end{aligned} \tag{6}$$

where the heat transfer coefficient between the two surfaces is a constant according to assumption (4). In Equation (6), the axial position of the displacer Z was replaced with a function of z and t via Equation (1). Obviously the initial conditions are not necessary for the solution of the cyclic steady state.

Solution

Equations (2) and (3) can be solved simultaneously with the boundary conditions (4), (5) and (6). For a periodic variation of temperatures, the equations can be solved analytically by introducing a complex temperature[6]. In this problem, a complex temperature is defined in relation to a physical temperature as

$$T \equiv Re(\theta) \tag{7}$$

for both the displacer and the cylinder.

Equations (2), (3), (4), (5) and (6) can be expressed in complex form, simply by replacing T with complex temperature θ and by replacing $\cos(\omega t)$ with $\exp(i\omega t)$. The two unknown variables are assumed to be

$$\theta_1(t,x_1,z) = T_0 + \Gamma\left[z - \frac{S}{2}\exp(i\omega t)\right] + \psi_1\exp(i\omega t) \tag{8}$$

$$\theta_2(t,x_2,z) = T_0 + \Gamma z + \psi_2\exp(i\omega t) \tag{9}$$

where ψ_1 and ψ_2 are complex. When the complex temperatures are substituted into the governing equations and the boundary conditions, the $\exp(i\omega t)$ factors drop out and the equations are reduced to algebraic equations. The two complex temperatures can then be determined and converted to the real temperatures by taking the real part of the complex temperatures as follows

$$\begin{aligned} T_1(t,x_1,z) = {} & T_0 + \Gamma\left[z - \frac{S}{2}\cos\omega t\right] \\ & + T_{10}\exp\left[-\sqrt{\frac{\omega}{2\alpha_1}}x_1\right] \\ & \times\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha_1}}x_1 + \frac{\pi}{4} - \phi\right) \end{aligned} \tag{10}$$

$$\begin{aligned} T_2(t,x_2,z) = {} & T_0 + \Gamma z + T_{20}\exp\left[-\sqrt{\frac{\omega}{2\alpha_2}}x_2\right] \\ & \times\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha_2}}x_2 + \frac{5\pi}{4} - \phi\right) \end{aligned} \tag{11}$$

where

$$T_{10} = \Gamma\frac{S}{2}\frac{a_2}{\sqrt{\left(\frac{a_1+a_2}{\sqrt{2}}\right)^2 + \left(\frac{a_1+a_2}{\sqrt{2}} + a_1a_2\right)^2}} \tag{12}$$

$$T_{20} = \Gamma\frac{S}{2}\frac{a_1}{\sqrt{\left(\frac{a_1+a_2}{\sqrt{2}}\right)^2 + \left(\frac{a_1+a_2}{\sqrt{2}} + a_1a_2\right)^2}} \tag{13}$$

$$\phi = \tan^{-1}\left(\frac{\frac{a_1+a_2}{\sqrt{2}} + a_1a_2}{\frac{a_1+a_2}{\sqrt{2}}}\right) \tag{14}$$

In Equations (12), (13) and (14), a_1 and a_2 are dimensionless numbers, defined by the wall properties, the heat transfer coefficient and the angular speed, as

$$a_1 \equiv \frac{k_1}{h}\sqrt{\frac{\omega}{\alpha_1}} \tag{15}$$

and

$$a_2 \equiv \frac{k_2}{h}\sqrt{\frac{\omega}{\alpha_2}} \tag{16}$$

respectively.

With the temperature of the displacer wall obtained, the shuttle heat transfer can now be calculated. At an arbitrary axial position, the shuttle heat transfer can be defined as the net enthalpy flow rate from the high to the low temperature side, or in the negative z direction. A cycle-averaged net enthalpy flow rate of the displacer wall can be obtained by integrating over a period and the displacer wall area

$$Q = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^{\infty}\rho_1c_1T_1\cdot\left(\frac{\omega S}{2}\sin\omega t\right)\cdot\pi D\mathrm{d}x\cdot\mathrm{d}t \tag{17}$$

where the parentheses denote the velocity of displacer in the negative z direction and $\pi D\mathrm{d}x$ denotes the infinitesimal area of the displacer wall according to assumption (6). The result of integrating Equation (17) is

$$Q = \frac{\pi DS}{4}(\omega k_1\rho_1c_1)^{1/2}\,T_{10}\sin\phi \tag{18}$$

where T_{10} and ϕ are defined in Equations (12) and (14), respectively. It can be noted that the shuttle heat transfer rate is independent of z with a constant temperature gradient.

Results and discussion

The validity of the new analytical expression for shuttle heat transfer, Equation (18), could be confirmed in several ways. First, if the heat capacity of the cylinder is much greater than that of the displacer, then

$$a_1 \ll a_2 \tag{19}$$

Then the temperature of the cylinder, Equation (11), is a function of z only, because T_{20} approaches zero in Equation (13). Thus the temperature of the cylinder does not oscillate in this case, as expected. And since

$$T_{10} = \frac{\Gamma S}{\sqrt{2}}\frac{1}{\sqrt{1 + (1 + \sqrt{2}a_1)^2}} \tag{20}$$

and

Table 1 Specifications of shuttle heat transfer system

Displacer	Material	Bakelite
	Diameter, D	100 mm
	Stroke, S	12 mm
	Frequency, $\omega/2\pi$	1 Hz
	Thickness	2 mm
Cylinder	Material	Stainless steel
	Thickness	4 mm
	Temperature gradient, Γ	1 K mm^{-1}
Gap	Gas	Helium
	Pressure/temperature	18 atm/150 K

$$\phi = \tan^{-1}(1 + \sqrt{2}a_1) \tag{21}$$

Equation (18) becomes the shuttle heat transfer, neglecting the temperature oscillation of the cylinder[5].

The new expression could also be compared with the existing data for a specific situation. The materials and physical dimensions for this calculation are given in *Table 1*. The shuttle heat transfer rates were calculated for a variable gap between the displacer and the cylinder using existing methods and the new method. In all cases, the heat transfer coefficient is obtained using

$$h = \frac{k_g}{\delta} \tag{22}$$

where k_g and δ are the thermal conductivity of gas and the gap between the walls, respectively.

Figure 2 shows the shuttle heat transfer as a function of the gap between the walls for four different cases. Case 1 and case 2 of Nishio *et al.*[5] indicate the results without and with temperature oscillation of the cylinder, respectively. In every case, the shuttle heat transfer decreases as the gap increases. The reason for this is that for a larger gap the temperature difference and the phase shift of the oscillation between the walls are smaller.

The shuttle heat transfer from Zimmerman *et al.* is inversely proportional to the gap distance and is much over-estimated due to the infinite heat capacity of the two walls. The result from Radebaugh *et al.*[3] has greater values of shuttle heat transfer at small gaps and lower values at large gaps. The result from case 1 of Nishio *et al.* has greater values than those from case 2 and from Equation (18). The cause of the difference is obvious; the phase shift from the temperature oscillation of the displacer to that of the cylinder was over-estimated due to the heat capacity of the cylinder being infinite.

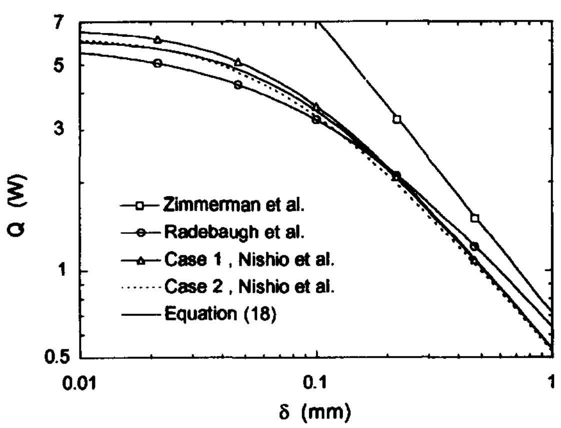

Figure 2 Comparison of current and existing expressions for shuttle heat transfer as a function of gap distance between walls

Fairly good agreement is observed between case 2 of Nishio *et al.* and Equation (18). The slight difference is due to the assumption of Nishio *et al.* that the phase shift of the surface temperature oscillation can be neglected[5]. Clearly, Equation (18) is much more useful than the calculation of Nishio *et al.*, because the former is an exact solution that satisfies the governing equations and the boundary conditions while the latter is an approximate solution with several simplifying assumptions. In addition to the accuracy, it is also obvious that Equation (18) is simpler to use.

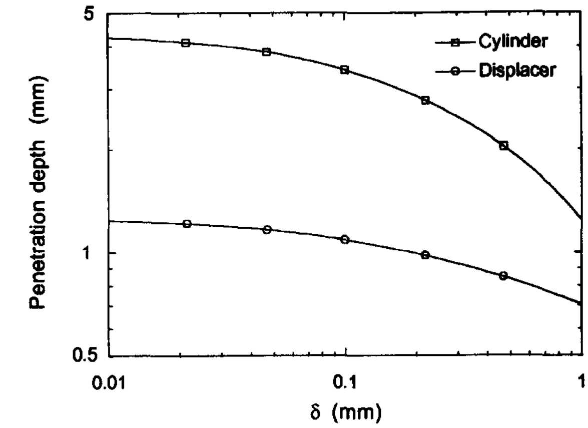

Figure 3 Thermal penetration depths of displacer and cylinder

As mentioned earlier, the validity of Equation (18) should be checked by seeing whether assumption (5) was justified. *Figure 3* shows the thermal penetration depth as a function of the gap distance between the walls, for both the displacer and the cylinder. In the calculation, the same dimensions as given in *Table 1* were used, and the thermal penetration depth is defined as the distance from the surface to the point where the amplitude of the temperature oscillation is 1% of the amplitude of the surface temperature oscillation. In this specific problem, the displacer penetration depth is always smaller than the wall thickness for all values of δ. On the other hand, the cylinder penetration depth is smaller than the wall thickness only when the gap distance is greater than $\approx$0.02 mm, which is a very small value in practice. It is concluded that Equation (18) is applicable to most real situations, but the validity should be checked with the penetration depth.

Figure 4 shows the effect of frequency of oscillation on

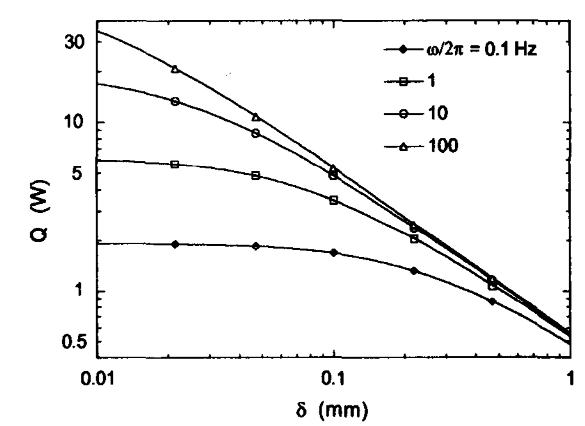

Figure 4 Shuttle heat transfer as a function of gap distance between walls for various oscillating frequencies

the shuttle heat transfer of Equation (18), given that assumption (5) is satisfied. In the conditions of *Table 1*, heat transfer increases with frequency for the same gap. At lower frequencies, the shuttle heat transfer increases but becomes saturated as the gap decreases. At higher frequencies, however, it continuously increases for the given conditions. It can be noticed that the shuttle heat transfer is independent of frequency at a large gap distance. This behaviour is consistent with previous reports.

Conclusions

A new analytical solution was obtained for the shuttle heat transfer rate, including the wall properties of both the displacer and the cylinder. Although the numerical results of the solution do not differ much in specific calculations from those of existing approximate methods, the new solution is superior in that it is mathematically exact and simple to use. The results are expected to be directly applicable to many cryogenic systems where shuttle heat transfer is significant.

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